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NATURAL CONVECTION IN A LONG RECTANGULAR CAVITY

K. P. Morgunov, T. Yu. Morgunova, and V. A. Misyura UDC 536.25

We discuss an approximate analytical method of calculating the parameters of the motion of a gas in a long cavity induced by the presence of a heated vertical wall. Assuming the flow is plane-parallel and the longitudinal temperature gradient in the central region of the flow is constant, we obtain analytical expressions for the velocity and temperature profiles. We use the law of conservation of energy in integral form to match the solution in the central region with the end regions near the walls, and thereby obtain the flow parameters without considering the structure of the flow in the end regions.

Introduction. The structure of the flow in a closed cavity containing a gas is determined by the temperature boundary conditions on the cavity walls. Previous studies of natural convection in cavities have been concerned mainly with small to moderate values of the ratio of the horizontal dimension L of the cavity to its vertical dimension H and comparatively little attention has been paid to flow in long horizontal cavities in the presence of a temperature gradient along the axis of the cavity. In [1-3] a two-dimensional rectangular closed cavity was considered whose length L was much larger than its height H, while the vertical walls were maintained at different constant temperatures. It was assumed that at a certain distance from the heated wall the parameters describing the flow vary much more rapidly in the transverse direction than along the cavity axis. With this assumption a relatively simple solution of the Navier–Stokes equations in the Boussinesq approximation can be obtained. We will assume that a solution of this type is correct for gas flow in a region sufficiently far from the vertical walls; this region is called the central flow region. The basic problem is to explain the effect of the conditions on the vertical walls of the cavity on the form of the solution in the central region.

We discuss below an approach to the matching of the flow in the central region to the flow near the vertical walls. The method is based on the integral conservation laws of continuum mechanics.

We consider a two-dimensional rectangular closed cavity containing a gas. The length L of the cavity is much larger than the height H (Fig. 1). The horizontal walls of the cavity are assumed to be adiabatic. One of the vertical walls is isothermal with temperature T_0. Two types of conditions are considered for the other vertical wall: the wall is isothermal with temperature T_H, greater than T_0; the specific heat flux q through the wall is specified.

The equations describing the steady laminar flow of a viscous incompressible liquid or gas in a horizontal cavity are, in dimensionless variables:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0; \tag{1}$$

$$\frac{1}{\mathrm{Pr}}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}; \tag{2}$$

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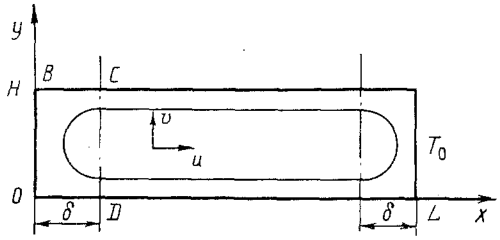

Fig. 1. Schematic diagram of the structure of the gas flow in a long cavity.

$$\frac{1}{\mathrm{Pr}}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mathrm{Ra}\,T+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}; \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}. \tag{4}$$

Here we have used the usual Boussinesq approximation for the density. The dimensionless variables are constructed as follows:

$$x=\frac{x^*}{H};\ y=\frac{y^*}{H};\ u=\frac{u^*}{a}H;\ v=\frac{v^*}{a}H;\ p=\frac{p^*H^2}{\rho_0 a\nu}.$$

An asterisk denotes a dimensional variable. The scale factor for the temperature depends on the boundary condition on the heated wall. In the case of an isothermal wall the dimensionless temperature is $T = (T^* - T_0)/(T_H - T_0)$. In this case the Rayleigh number Ra in (3), which characterizes the intensity of natural convection, has the form $\mathrm{Ra} = \beta g H^3(T_H - T_0)/a\nu$. If the heat flux is specified on the vertical wall, then the dimensionless temperature will be: $T = \lambda(T^* - T_0)/Hq$ and in (3) it will be necessary to use the modified Rayleigh number $\mathrm{Ra}^* = \beta g H^4 q/a\nu\lambda$.

The boundary conditions will have the form

$$u=v=0;\ \frac{\partial T}{\partial y}=0 \quad \text{at} \quad y=0;\ 1; \tag{5}$$

$$u=v=0;\ T=0 \quad \text{at} \quad x=L/H.$$

At $x = 0$ we have $T = 1$ for a constant wall temperature and $\partial T/\partial x = 1$ for a constant heat flux on the wall.

We split up the flow region into three parts: the central region and the end regions near the two vertical walls (Fig. 1).

Numerical calculations confirm the conclusion of [1, 3] that in the central flow region the streamlines are parallel to the horizontal walls and the temperature gradient along the x axis is constant. Hence, in the central region

$$v=0;\ \frac{\partial T}{\partial x}=K_1=\mathrm{const}. \tag{6}$$

We use condition (6) in solving the system (1)-(4) in the central flow region. First we eliminate the pressure from these equations by differentiating (2) with respect to y, differentiating (3) with respect to x, and subtracting the two resulting equations. Then with the help of (6), the system of equations (1)-(4) can be written in the form

$$\frac{\partial^3 u}{\partial y^3}-\mathrm{Ra}\,K_1=0; \tag{7}$$

$$uK_1=\frac{\partial^2 T}{\partial y^2}. \tag{8}$$

To solve (7) one more condition must be imposed on u [in addition to (5)]. For this condition we use the integral conservation of mass condition in a cross section of the cavity

$$\int_0^1 u dy = 0. \tag{9}$$

The solution of (7) subject to the conditions (5) and (9) will be

$$u = \mathrm{Ra}\, K_1 \left(\frac{y^3}{6} - \frac{y^2}{4} + \frac{y}{12}\right). \tag{10}$$

With the help of (5) and (10), we obtain from (8) an expression for the temperature in the central region:

$$T = \mathrm{Ra}\, K_1 \left(\frac{y^5}{120} - \frac{y^4}{48} + \frac{y^3}{72}\right) + K_1 x + K_2. \tag{11}$$

The parameters K_1 and K_2 in (10) and (11) are determined by requiring that the temperature and velocity distributions in the central region be consistent with the corresponding distributions in the end regions.

The following procedure was used in [3] to match the solution in the central region to the velocity and temperature distributions in the end regions and to thereby determine the constants K_1 and K_2. The differential equations of motion and energy (2)-(4) were integrated with respect to x and y over the extent of the end region [the pressure was first eliminated from (2) and (3)]. Then acceptable velocity and temperature distributions in the end regions were chosen in a rather arbitrary way. The necessary conditions are: these distributions must be consistent with the solutions for the central region on the matching boundaries; the boundary conditions on the solid walls must be satisfied, and the equation of continuity at any point inside the region must be satisfied. By substituting the chosen distributions in the end regions into the integral conditions, two expressions are obtained for the two unknown constants.

We use the general integral form of the energy equation to construct the conditions in the end regions. In this method one avoids the basic difficulty in the consistency procedure described above: the choice of velocity and temperature distributions for the end regions.

We introduce the dimensionless length of an end region δ (Fig. 1). We define δ as the part of the cavity near a vertical wall inside which the solution for the central region (10) and (11) becomes incorrect.

The general integral form of the energy equation (first law of thermodynamics) is [4]:

$$\int_{\Sigma} \rho \left(\frac{W^2}{2} + U\right) W_n d\sigma = \int_{V} \left(\mathbf{FW} + \frac{dq_{\text{mass}}}{dt}\right) \rho d\tau + \int_{\Sigma} (\mathbf{p}_n \mathbf{W} - q^*)\, d\sigma. \tag{12}$$

Here Σ is the surface enclosing the arbitrary volume V and vectors are denoted by bold-face letters.

Note that if everything were perfectly consistent, the integral form of the energy equation could be deduced from (1)-(4) and the Boussinesq approximation. However, the error in using the energy equation in the form (12) will be of the same order as the error introduced into the Navier—Stokes equations by the Boussinesq approximation. Because of the approximate nature of our analysis, we neglect this error. We choose the contour OBCD (Fig. 1) as the contour Σ for the end region next to the heated wall. Note that the velocity W vanishes on all segments of the contour except CD. On CD the velocity and temperature correspond to the values for the central region and are determined from (10) and (11). The pressure distribution in the central region (and hence on the line CD) can be determined from (2) and (3). Substituting (10) for the velocity and (11) for the temperature into (2) and (3), we have

$$p = \mathrm{Ra}^2 K_1^2 \left(\frac{y^6}{720} - \frac{y^5}{240} + \frac{y^4}{288}\right) + \mathrm{Ra}\, K_1 x \left(y - \frac{1}{2}\right) + \mathrm{Ra}\, K_2 y.$$

The internal energy of the gas U in (12) is defined as U = cT. The term dq_{mass}/dt represents the flow of energy per unit time in the volume due to mass sources. In our case it is equal to zero.

We take the force of gravity as the body force F. Then

$$\int_V \rho \mathbf{F} \mathbf{W} d\tau = Mg\mathbf{W}^*, \tag{13}$$

where M is the mass of the gas inside the volume V and $\mathbf{W}^*$ is the velocity of the center of mass of the volume V.

The quantity $\mathbf{W}^*$ in the end regions is quite small and the integral (13) can be made negligibly small by choice of the length of the end region δ. The choice of δ will be discussed below.

The external flow of energy into the volume is $q^* = \lambda(T_H - T_\delta^*)/\delta^*$ in the case when the temperature of the wall is specified (T_δ^* is the temperature of the gas in the cross section $x^* = \delta^*$) or $q^* = q$ in the case when the heat flux is specified at the wall.

Combining these remarks with the help of (12), we obtain

$$\mathrm{Ra}^2 K_1^3 - 504\,\mathrm{Ra}\,\mathrm{Em}\left(\frac{\mathrm{Ra}\,K_1^3}{1440} + K_1K_2 + K_1^2\delta\right) = \frac{362880}{\delta}\left(1 - \frac{\mathrm{Ra}\,K_1^2}{1440} - K_1\delta - K_2\right) \tag{14}$$

for an isothermal wall, and

$$\mathrm{Ra}^{*2} K_1^3 - 504\,\mathrm{Ra}^*\,\mathrm{Em}\left(\frac{\mathrm{Ra}^*\,K_1^3}{1440} + K_1K_2 + K_1^2\delta\right) = 362\,880 \tag{15}$$

in the case where the heat flux is specified at the wall. The relations (14) and (15) involve the dimensionless parameter $\mathrm{Em} = \beta gH/c$.

We obtain a second condition for the constants K_1 and K_2 by constructing an expression analogous to (12) for the other end region. For the vertical wall with constant temperature $T_{x=L/H}$ we will have

$$\mathrm{Ra}^2 K_1^3 - 504\,\mathrm{Ra}\,\mathrm{Em}\left(\frac{\mathrm{Ra}\,K_1^3}{1440} + K_1K_2 + K_1^2\frac{L}{H} - K_1^2\delta\right) = \frac{362880}{\delta}\left(\frac{\mathrm{Ra}\,K_1^2}{1440} + K_1\frac{L}{H} - K_1\delta + K_2\right). \tag{16}$$

Hence the two unknown constants K_1 and K_2 are found from the system of equations (14) and (16) or (15) and (16) as functions of Ra, Em, L/H, and the length of the end zone δ.

We estimate the quantity δ. In [3] additional conditions were imposed on the flow in order to determine the length of the end zone. An example is the value of the temperature at the center of the cavity (the point with coordinates L/2H; 1/2). However, recalling the definition of δ as the dimension of the region in which the solution in the central part of the cavity becomes incorrect, we note that the accuracy of the approximation of the solution obtained above to the true solution will be greater, the larger the value of δ ($0 < \delta < L/2H$). On the other hand, the condition that the integral of the body forces (13) be small imposes a restriction on δ. In contrast to [3], in our case δ is a free parameter of the problem, and is not determined as part of the solution.

We note that the solution obtained here is correct only in the case when the flow in the cavity is a counterflow, as described by (10). In connection with this it is interesting to analyze qualitatively the changes in the nature of the flow with changes in the parameters determining the flow, i.e., the numbers Ra and Pr and the ratio of the sides of the cavity L/H.

For small Rayleigh numbers the motion in a long closed cavity is a counterflow, as considered in the present paper, and its intensity is small in view of the smallness of the Rayleigh number. With increasing Rayleigh number boundary layers form on the vertical walls. With a further increase of the Rayleigh number more and more momentum arrives in the central region from the end regions and in turn from the rapidly moving boundary layers on the vertical walls. Heat obtained by the gas from the heated vertical wall feeds into the upper branch of the counterflow in the central region. This heat is then transferred by thermal conduction

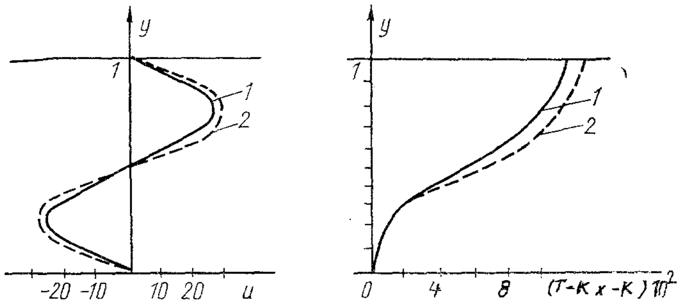

Fig. 2. Horizontal component of the velocity and temperature in the central region for Ra = 10^4: 1) from (10) and (11); 2) numerical calculation.

from the hot branch to the cold lower branch of the counterflow. This situation persists until the necessary amount of heat is transferred by thermal conduction from the upper branch of the counterflow to the lower branch. A dimensional analysis carried out by Gill [5] showed that the heat feeding into the central region due to natural convection is a quantity of order $k\Delta T\ Ra^{1/4}$, whereas the heat drawn off by thermal conduction is of order $k\Delta T\ L/H$. Hence, the counterflow can exist in the central region as long as we have the relation $Ra < (L/H)^4$.

We note that this analysis is correct for the case when the flow is determined by the viscous forces, i.e., the Prandtl number should not be too small ($Pr \geq 1$).

It was confirmed in [1] that with increasing L/H the Rayleigh number, which determines the upper boundary of existence of this flow regime, also increases.

The nature of the flow changes for still larger Rayleigh numbers. Boundary layers are formed on the horizontal thermally insulated walls. Flow practically ceases in the rest of the central region and a stagnent zone is formed in which the temperature varies linearly with height and is constant along the length of the cavity. This is the so-called boundary-layer regime [6], in which the nature of the flow differs qualitatively from that considered here; in particular, the velocity distribution in the form (10) no longer corresponds to the actual flow structure in this case.

The Prandtl number determines the effect of the viscous forces on the flow. For small Prandtl numbers ($Pr \ll 1$) the thickness of the viscous boundary layers on the vertical walls of the cavity is much less than the thickness of the thermal boundary layer. This affects the flow in the entire cavity. It was established in [7] that when $Pr < 0.2$ a hydrodynamic instability arises in a cavity with thermally insulated horizontal walls and the counterflow structure of the flow breaks down and multi-cellular flow arises in the cavity.

We solved the complete system of differential equations (1)-(4) numerically in order to assess the adequacy of our analytical method. Difference equations were constructed and solved with the help of a modification of the Patankar–Spolding method [8]. We used the natural variables (velocity, temperature, and pressure), which facilitates the approximation of the boundary conditions and the interpretation of the results. Calculations were done for relative cavity lengths L/H = 1-10, Rayleigh numbers Ra = 10-10^6, and Prandtl numbers Pr = 0.5-5. The results of the numerical calculations are shown in Figs. 2 and 3. The results confirm the qualitative picture of the flow and closely correspond with the results of [1-3]. The order of magnitude of the quantity δ can be determined from the numerical results. We see from Fig. 3 that the calculations give $\delta \approx 0.5$-0.7. Hence, the length of the end zones can be estimated to a sufficient degree of accuracy by putting $\delta = 1$.

The deviations between the numerical results and the approximate analytical dependences are 8-10% for Rayleigh numbers in the interval 10^2-10^6. This can be interpreted as the error in the approximate method. The deviations increase markedly with further increase in the Rayleigh number, since there is a qualitative change in the structure of the flow.

The parameter Em appears in the relations (14)-(16) for the constants K_1 and K_2. The quantity Em is an analog of the Eckert number for free convection and characterizes the ratio of the kinetic energy of the flow to the heat transferred to the flow from the heated wall.

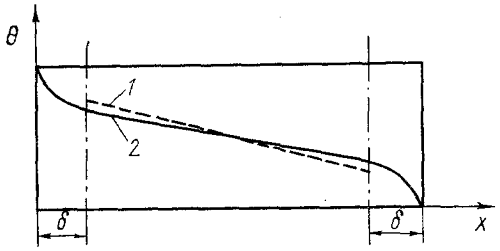

Fig. 3. Temperature distribution along the axis of a long cavity for $Ra = 10^4$: 1) from (11); 2) numerical calculation.

Conclusion. The approximate analytical method discussed here of calculating the flow of gas in a long horizontal cavity is correct for the parameter space in which a counterflow with a cubic velocity profile (10) holds in the central part of the cavity. Comparison of the structure of the flow obtained here with the results of [1-3] and with the results of the numerical calculations confirms the adequacy of our method.

It is interesting also to note the absence of the effect of the Prandtl number on the flow parameters in our analytical solution. This is a consequence of the assumption (6), i.e., the flow in the central region is fully developed (does not depend on x) and in this case the inertial terms in (1)-(4) vanish. The numerical solution confirmed that the solution is relatively independent of Prandtl number in the intervals of Prandtl number and Rayleigh number considered here.

We note that in the case of fully developed, steady-state flow considered here, the nature of the flow is determined by the conditions on the surfaces bounding the ends of the region, assuming that the volume integral (13) is small. Hence, it is not necessary to determine the qualitative or quantitative structure of the flow inside these regions. The flow in channels and cavities with complicated end regions can be studied with this approach if the flow conditions are known on the boundaries of these regions. Since δ is a free parameter of the problem, we can give it different values for the regions near the hot and cold walls, which widens the region of applicability of the method.

The results obtained here can be applied to different fields of engineering mechanics, in the calculation of heat-transfer devices, solar energy systems, and so on.

NOTATION

a, thermal diffusivity; c, heat capacity of the gas; Em, dimensionless parameter; F, body force; g, acceleration of gravity; H, height of the cavity; K_1 and K_2, dimensionless constants determining the velocity and temperature profiles; L, cavity length; M, mass; p, pressure; Pr, Prandtl number; q, heat flux; Ra, Rayleigh number; T, gas temperature; T_0, temperature of the cold wall; T_H, temperature of the heated wall; u, horizontal component of the gas velocity; U, internal energy of the gas; v, vertical component of the gas velocity; V, arbitrary volume of the gas; W, gas velocity; x, y, horizontal and vertical coordinates; β, volume coefficient of thermal expansion of the gas; δ, length of the end regions; λ, thermal conductivity; Σ, surface enclosing the arbitrary volume V; ρ, gas density; ν, kinematic viscosity of the gas.

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